INSIDERS AND THEIR FREE LUNCHES: THE ROLE OF SHORT POSITIONS

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ABSTRACT. Given a stock price process, we analyse the potential of arbitrage by insiders in a context of short-selling prohibitions. We introduce the notion of minimal supermartingale measure, and we analyse its properties in connection to the minimal martingale measure. In particular, we establish conditions when both fail to exist. These correspond to the case when the insider's information set includes some non null events that are perceived as having null probabilities by the uninformed market investors. These results may have different applications, such as in problems related to the local risk-minimisation for insiders whenever strategies are implemented without short selling.

1. INTRODUCTION

Given a stock price process, we analyse the potential of arbitrage by insiders, when they chose not to sell short. Short-selling stocks when having access to a value-destroying event before it is known to the public leads to profits, but insiders that are already detaining the stocks may profit without engaging in selling short. Selling (and buying) of company shares by people who have special information because they are involved with the company is illegal in many jurisdictions, whenever the information is non-public and material. Insider trading scandals are nevertheless regularly on the front page of newspapers. For instance, the US justice department is currently investigating alleged insider trading by US lawmakers who sold stocks just before the coronavirus pandemic sparked a major market downturn. Rarely short selling is part of the insiders' strategies in these scandals. Short selling transactions receive additional attention from supervisory authorities and it is possible that many insiders refrain from such strategies for their trades to remain undetected. It is not clear nevertheless, from both a practical and purely theoretical perspective if an insider that does not detain the stock has the possibility to first buy and then sell the stock in a profitable manner, thus exploiting the informational advantage without selling short. The aim of this paper is to propose some first elements in tackling this question. For the notion of no arbitrage profits, we consider the setting of no free lunch with vanishing risks with short sales prohibitions (NFLVRS) introduced in Pulido [33], a restriction of the classical notion of no free lunch with vanishing risks (NFLVR) by Delbaen and Schachermeyer [13], [15]. Based on previous work by Jouini and Kallal [25], Fritelli [19], Pham and Touzi [32], Napp [28] and Karatzas and Kardaras [26], the paper by Pulido [33] establishes important properties of price processes under short sale prohibitions namely the equivalence between (NFLVRS) and the existence of an equivalent supermartingale measure for the price processes.

There is a very vast literature studying arbitrage possibilities and charactering no-arbitrage models, for different types of inside information and for different notions of arbitrage, sometimes weaker that no free lunch with vanishing risks. Usually, a model with respect to a given information set (filtration) is specified on a probability space, representing the financial market such as common investors perceive it, and the insider information is introduced as a strictly larger information set, via an enlargement of the original filtration. A fundamental question in this context is whether the additional information allows for arbitrage profits. We mention the early work by

Grorud and Pontier [20], Amendinger et al. [4], Imkeller [23] and subsequent work by Hillairet [21], Acciaio et al. [1], Aksamit et al. [2], Fontana et al [18], Chau et al [9], among athers. In a general setting, the situations where the additional information of an insider is insufficient for obtaining profits of arbitrage in the form of no free lunch with vanishing risk, were characterised in the literature and linked to the so-called (H) hypothesis holding between the asset price filtration and the larger filtration containing the additional information (see Blanchet-Scalliet and Jeanblanc [6] and Coculescu et al. [11]).

The present paper contributes to the literature on insider trading by studying specifically the short-sales restrictions. Within the two-filtration setting above described, with a filtration larger than the other, and where a stock price is an adapted process for both filtrations, we shall assume that no free lunch with vanishing risks prevails for the common investors having access only to the smallest filtration. We will then exclusively focus on the no arbitrage conditions in the larger filtration, that is, when considering the insider information. As mentioned above, the existence of a supermartingale measure for the stock price is equivalent to (NFLVRS) (Pulido [33]). After we introduce the notion of minimal supermartingale measure, we analyse some of its connections with the minimal martingale measure (Section 3). We then provide an additional analysis when assuming the smaller filtration has the predictable representation property with respect to a continuous martingale and assuming that the (H) hypothesis is satisfied under a probability measure such that the original one is absolutely continuous to it (Section 4); when such measures are not equivalent, we prove the failure of both the minimal martingale measure and the minimal supermartingale measure to exist.

2. The mathematical setting

A probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is fixed. We consider a financial market where a risky asset (e.g. a stock) is traded, in addition to a risk-free asset. We consider a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions and such the price process, denoted by $X = (X_t)_{t\geq 0}$ is adapted. To keep the message simple, we assume that X is a $(\mathbb{F}, \mathbb{P})$ local martingale and that the price of the risk-free asset is constant. Alternatively, one could consider the discounted asset price and assume the existence of an equivalent martingale measure for it in the filtration $\mathbb{F}$. We have thus a standard arbitrage free market whenever the filtration $\mathbb{F}$ summarises the available information. We shall call *common investors* the $\mathbb{F}$ informed investors.

We now take the point of view of a particular agent, that we name the *insider*. We assume that the insider has access to some information that can potentially impact the risky asset price, should this information be publicly released. The insider information is modelled by a filtration $\mathbb{G}$ strictly larger than $\mathbb{F}$, i.e., such that

$$\mathcal{F}_t \subset \mathcal{G}_t\ ,\ t \geq 0 \text{ and } \mathcal{F}_\infty \neq \mathcal{G}_\infty.$$

We do not assume any particular structure of the filtration $\mathbb{G}$, except being strictly larger than $\mathbb{F}$ and satisfying the usual assumptions.

Before considering the question of whether the insider can have free lunches, one should clarify the available trading strategies for the insider. The insider is supposed to invest in the risky and risk free asset based on her information. We are interested here in two alternative situations: the one where the insider may sell short the risky asset and the one where she is constrained to have only long positions in the risky asset, i.e., short sales are prohibited. Consequently, we define the following sets for $t \geq 0$:

$\mathcal{M}_t(\mathbb{G},\mathbb{P}) := \{\mathbb{Q} \sim \mathbb{P} \text{ on } \mathcal{G}_t \mid X_{\cdot\wedge t} \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ local martingale}\}$ and $\mathcal{M}(\mathbb{G},\mathbb{P}) := \mathcal{M}_\infty(\mathbb{G},\mathbb{P})$;

$\mathcal{S}up_t(\mathbb{G},\mathbb{P}) := \{\mathbb{Q} \sim \mathbb{P} \text{ on } \mathcal{G}_t \mid X_{\cdot\wedge t} \text{ is a } (\mathbb{G},\mathbb{Q}) \text{ supermartingale}\}$ and $\mathcal{S}up(\mathbb{G},\mathbb{P}) := \mathcal{S}up_\infty(\mathbb{G},\mathbb{P})$.

In this note we intend to analyse from the insider's perspective the connections between arbitrages involving short selling the risky asset and arbitrages that exclude short selling the risky asset. The no arbitrage conditions we study are relative to the insider filtration $\mathbb{G}$, and they are formally the following:

(a) the no free lunch with vanishing risks (NFLVR) condition of Delbaen and Schachermayer [13], in the case where short positions are permitted along with the long positions in the assets. In our setting, (NFLVR) is equivalent to: $\mathcal{M}(\mathbb{G},\mathbb{P}) \neq \emptyset$.

(b) the no free lunch with vanishing risk under short sales prohibition (NFLVRS), a condition that was introduced in Pulido [33]. In our setting, by short sales prohibitions we mean that long positions in the risky asset are permitted, along with short and long in the risk-free asset, and in this case (NFLVRS) is equivalent to: $\mathcal{S}up(\mathbb{G},\mathbb{P}) \neq \emptyset$ (cf. Theorem 3.9 in [33]).

Notation. For any $\mathbb{G}$-adapted process Y, we denote ${}^oY$ its optional projection onto the filtration $\mathbb{F}$ with respect to the measure $\mathbb{P}$; whenever Y is increasing, Y^o stands for its $(\mathbb{F},\mathbb{P})$-dual optional projection. For a semimartingale Z with $Z_0 = 0$, $\mathcal{E}(Z)$ denotes the Doléans-Dade exponential.

Some results will be established in the particular settings where the process X has continuous sample paths, or $\mathbb{F}$ enjoys the predictable representation property, that is, we have a complete $\mathbb{F}$ market. We label these particular settings as follows:

(C) The process X is a continuous $(\mathbb{F},\mathbb{P})$ martingale.

$\mathbb{F}$-(PRP) The local martingale X satisfies the predictable representation property with respect to the filtration $\mathbb{F}$, that is, any $\mathbb{F}$ local martingale vanishing at zero is equal to $H \cdot X$ for a suitable $\mathbb{F}$ predictable process H.

Importantly, we shall assume a certain structure for the decomposition of the stochastic process X in the insider filtration $\mathbb{G}$, namely that X is a $(\mathbb{G},\mathbb{P})$ semimartingale of the form:

$$X = X_0 + M + \int \alpha d\langle M\rangle \tag{1}$$

for some $(\mathbb{G},\mathbb{P})$ local martingale M and a $(\mathbb{G},\mathbb{P})$ predictable process α having a null $(\mathbb{F},\mathbb{P})$ optional projection (that is, ${}^o\alpha \equiv 0$).

Remark 2.1. *We remark that the process α is not uniquely defined on the intervals where $d\langle M\rangle = 0$. For simplicity, we shall assume α to be constantly zero on these intervals.*

The choice of assuming a decomposition of X as proposed in (1) is rooted in the result below:

Theorem 2.2. *Suppose that at least one of the following two holds true:*

1. $\mathcal{M}(\mathbb{G},\mathbb{P}) \neq \emptyset$
2. *(C) and $\mathcal{S}up(\mathbb{G},\mathbb{P}) \neq \emptyset$.*

Then X is a $(\mathbb{G},\mathbb{P})$ a semimartingale which decomposes as in (1).

Proof. In the case where $\mathcal{M}(\mathbb{G},\mathbb{P}) \neq \emptyset$ holds, the decomposition in (1) is due to Ansel and Stricker [5] and Schweizer [35] for a $\mathbb{G}$ predictable process α, but without the requirement that ${}^o\alpha \equiv 0$. The same holds when $\mathcal{S}up(\mathbb{G},\mathbb{P}) \neq \emptyset$, as proved by Coculescu and Jeanblanc [10] (note that should

X be a discontinuous price process the decomposition above is not automatic). It remains to prove that process α has a null $(\mathbb{F},\mathbb{P})$ optional projection. This is a consequence of X being a $\mathbb{P}$-local martingale in the smaller filtration $\mathbb{F}$ (see Brémaud and Yor [7], Proposition 3). $\square$

Further analysis in Section 4 will again make appear the representation (1). We can say that this expression is therefore more general than it may appear at first sight. It is important to keep in mind that we assume that no free lunch with vanishing risk holds for common investors (that is in the smaller filtration $\mathbb{F}$), but we analyse the properties of the price process when some additional information is available for an insider, that is, in the filtration $\mathbb{G}$. Of course, without inside information, the decomposition of a price process will not have any sound justification to be such as in (1).

From Theorem 2.2, we may characterise a framework where the insider can have free lunches with vanishing risk without engaging in short sales:

Corollary 2.3. *Suppose that (C) holds. Then, if X does not have the representation (1), it follows that $\mathcal{M}(\mathbb{G},\mathbb{P})=\emptyset$ and $\mathcal{S}up(\mathbb{G},\mathbb{P})=\emptyset$.*

In Coculescu and Jeanblanc [10], examples are provided where the $(\mathbb{G},\mathbb{P})$ decomposition of X fails to satisfy (1), in the case where $\mathbb{G}$ is obtained by progressive enlargements of $\mathbb{F}$ with a random time. The corresponding arbitrage portfolios are also discussed. The current paper is intended to further analyse the question of (NFLVRS) within the case where the representation (1) for the stock price holds.

3. The minimal supermartingale measure: definition and some sufficient conditions for its existence

We use the setting from the previous section. Note nevertheless, that the condition ${}^o\alpha\equiv 0$ in the representation (1) is not necessary for establishing the results of in this section, but will play an important role in the next section. We introduce the following $(\mathbb{G},\mathbb{P})$ local martingales:

$$R_t:=\mathcal{E}_t\left(-\int_0^{\cdot}\alpha_u dM_u\right),\quad t\geq 0 \tag{2}$$

$$R_t^{(+)}:=\mathcal{E}_t\left(-\int_0^{\cdot}(\alpha_u)^+ dM_u\right),\quad t\geq 0. \tag{3}$$

We denote

$$\frac{d\mathbb{Q}^m}{d\mathbb{P}}\Big|_{\mathcal{G}_t}=R_t,$$

and if R is a strictly positive and uniformly integrable martingale, we have $\mathbb{Q}^m\in\mathcal{M}(\mathbb{G},\mathbb{P})$. Similarily, we denote

$$\frac{d\mathbb{Q}^{sup}}{d\mathbb{P}}\Big|_{\mathcal{G}_t}=R_t^{(+)},$$

and if $R^{(+)}$ is a strictly positive and uniformly integrable martingale we have $\mathbb{Q}^{sup}\in\mathcal{S}up(\mathbb{G},\mathbb{P})$ which can be checked using Girsanov's theorem. In general, $\mathbb{Q}^m$ and $\mathbb{Q}^{sup}$ are signed measures.

We recall below the notion of minimal martingale measure which first appeared in Schweizer [34] (see also Föllmer and Schweizer ([17]) for a survey of this very rich topic).

Definition 3.1. *When R is a strictly positive and square integrable $(\mathbb{G},\mathbb{P})$ martingale, we call $\mathbb{Q}^m\in\mathcal{M}(\mathbb{G},\mathbb{P})$ the minimal martingale measure for X.*

Symmetrically, let us introduce the following definition:

Definition 3.2. *The minimal supermartingale measure for X, when it exists, is a probability measure $\mathbb{Q}^{sup} \in \mathcal{S}up(\mathbb{G},\mathbb{P})$ that satisfies:*

$$\frac{d\mathbb{Q}^{sup}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} = R_t^{(+)} \in L^2(\mathbb{P}).$$

It can be shown that when X has continuous sample paths, the minimal supremartingale measure can be characterised as the unique solution of a minimisation problem involving the relative entropy with respect to $\mathbb{P}$, in a parallel manner to the result holding for the minimal martingale measure; in both cases the idea is to obtain a "minimal modification" of $\mathbb{P}$ that allows to reach the set of supermartingale versus martingale measures. The formal setting is given below.

Definition 3.3. *The relative entropy, $H(\mathbb{Q}|\mathbb{P})$, measuring the departure of a measure $\mathbb{Q}$ from a given measure $\mathbb{P}$, is defined as*

$$H(\mathbb{Q}|\mathbb{P}) = \begin{cases} \int \log \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{Q} & \text{if } \mathbb{Q} \ll \mathbb{P}. \\ +\infty, & \text{otherwise}. \end{cases}$$

The relative entropy H is nonnegative and $H(\mathbb{Q}|\mathbb{P}) = 0$ if and only if $\mathbb{Q} = \mathbb{P}$.

Theorem 3.4. *Suppose that (C) holds. Also we assume that $\mathbb{Q}^{sup} \in \mathcal{S}up_T(\mathbb{G},\mathbb{P})$ and $H(\mathbb{Q}^{sup}|\mathbb{P}) < \infty$. Then, the measure $\mathbb{Q}^{sup}$ is the unique solution of*

$$\text{Minimize } H(\mathbb{Q}|\mathbb{P}) - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\alpha_s^+)^2 d\langle X\rangle_s\right], \tag{4}$$

over all supermartingale measures $\mathbb{Q} \in \mathcal{S}up_T(\mathbb{G},\mathbb{P})$, satisfying the condition $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\alpha_s^+)^2 d\langle X\rangle_s\right] < \infty$.

Proof. Let $\mathbb{Q} \in \mathcal{S}up_T(\mathbb{G},\mathbb{P})$ and $G_t^{\mathbb{Q}} = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_t}$, where

$$G_t^{\mathbb{Q}} = G_0^{\mathbb{Q}} + \int_0^t \delta_s^{\mathbb{Q}} dM_s + L_t^{\mathbb{Q}}, \quad t \in [0,T].$$

In the above, $L^{\mathbb{Q}}$ is a $(\mathbb{G},\mathbb{P})$ martingale orthogonal to M and $\delta^{\mathbb{Q}}$ is some $\mathbb{G}$ predictable process.

Under $(\mathbb{G},\mathbb{Q})$, X possesses the following decomposition:

$$X_t = M_t^{\mathbb{Q}} + \int_0^t (\alpha_s + \beta_s^{\mathbb{Q}}) d\langle M\rangle_s$$

and $\beta^{\mathbb{Q}} = \frac{\delta^{\mathbb{Q}}}{G^{\mathbb{Q}}}$. Here, $M^{\mathbb{Q}} = M - \int \frac{1}{G^{\mathbb{Q}}} d\langle G^{\mathbb{Q}}, M\rangle = M - \int \beta^{\mathbb{Q}} d\langle M\rangle$ is a $(\mathbb{G},\mathbb{Q})$ martingale

As $\mathbb{Q}$ is a supermartingale measure, $\int(\alpha + \beta^{\mathbb{Q}})d\langle M\rangle$ is a decreasing process, meaning that for $t \in [0,T]$

$$(\alpha_t + \beta_t^{\mathbb{Q}})\mathbb{1}_{d\langle M\rangle_t > 0} \le 0. \tag{5}$$

We have that

$$\begin{aligned} H(\mathbb{Q}|\mathbb{P}) &= H(\mathbb{Q}|\mathbb{Q}^{sup}) + \int \log R_T^{(+)} d\mathbb{Q} \\ &= H(\mathbb{Q}|\mathbb{Q}^{sup}) + \int \left(-\int_0^T \alpha_s^+ dM_s - \frac{1}{2}\int_0^T (\alpha_s^+)^2 d\langle M\rangle_s \right) d\mathbb{Q} \\ &= H(\mathbb{Q}|\mathbb{Q}^{sup}) + \int \left(-\int_0^T \alpha_s^+ dM_s^{\mathbb{Q}} - \int_0^T \alpha_s^+ \beta_s^{\mathbb{Q}} d\langle M\rangle_s - \frac{1}{2}\int_0^T (\alpha_s^+)^2 d\langle M\rangle_s \right) d\mathbb{Q} \\ &= H(\mathbb{Q}|\mathbb{Q}^{sup}) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (-\alpha_s^+ \beta_s^{\mathbb{Q}} - \frac{1}{2}(\alpha_s^+)^2) d\langle X\rangle_s \right]. \end{aligned}$$

Therefore,

$$H(\mathbb{Q}|\mathbb{P}) - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\alpha_s^+)^2 d\langle X\rangle_s \right] = H(\mathbb{Q}|\mathbb{Q}^{sup}) - E^{\mathbb{Q}}\left[\int_0^T \alpha_s^+(\alpha_s + \beta_s^{\mathbb{Q}}) d\langle X\rangle_s \right].$$

and

$$H(\mathbb{Q}|\mathbb{P}) - \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\alpha_s^+)^2 d\langle X\rangle_s \right] \geq \min_{\mathbb{Q}} H(\mathbb{Q}|\mathbb{Q}^{sup}) - \max_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \alpha_s^+(\alpha_s + \beta_s^{\mathbb{Q}}) d\langle X\rangle_s \right],$$

where the min and max above are taken over the same set as in the problem (4); it can be checked that $\mathbb{Q}^{sup}$ belongs in this set.

We have that $\min_{\mathbb{Q}} H(\mathbb{Q}|\mathbb{Q}^{sup}) = 0$; the minimal value of 0 is obtained if and only if $\mathbb{Q} = \mathbb{Q}^{sup}$. Also, in view of the inequality (5), $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \alpha_s^+(\alpha_s + \beta_s^{\mathbb{Q}}) d\langle X\rangle_s\right] \leq 0$ and we can check that the maximum value of 0 is obtained taking $\beta^{\mathbb{Q}} = -\alpha^+$, that is, $G^{\mathbb{Q}} = R^{(+)}$, or $\mathbb{Q} = \mathbb{Q}^{sup}$. This proves that (4) equals 0 and this value is attained if and only if $\mathbb{Q} = \mathbb{Q}^{sup}$.

Let us now emphasise some connections between $\mathbb{Q}^m$ and $\mathbb{Q}^{sup}$:

Theorem 3.5. (a) *If the minimal martingale measure $\mathbb{Q}^m$ exists then, $\mathbb{Q}^{sup} \in \mathcal{S}up(\mathbb{G}, \mathbb{P})$.*

(b) *Suppose that the process α has a bounded number of strictly positive excursions. If $\mathbb{Q}^m \in \mathcal{M}(\mathbb{G}, \mathbb{P})$ then $\mathbb{Q}^{sup} \in \mathcal{S}(\mathbb{G}, \mathbb{P})$.*

Proof.

(a) We need to prove that if R is a strictly positive and square integrable $(\mathbb{G}, \mathbb{P})$ martingale then $R^{(+)}$ is a strictly positive and uniformly integrable $(\mathbb{G}, \mathbb{P})$ martingale. Let us introduce the following $(\mathbb{G}, \mathbb{P})$ local martingale, which is orthogonal to $R^{(+)}$:

$$R_t^{(-)} = \mathcal{E}_t\left(\int_0^{\cdot} (\alpha_u)^- dM_u \right), \quad t \geq 0,$$

so that:

$$R_t := R_t^{(+)} R_t^{(-)}.$$

The processes $R^{(+)}$ and $R^{(-)}$ are strictly positive local martingales: they satisfy $R_0^{(+)} = R_0^{(-)} = 1$ and can become negative or null only after a jump, i.e., one at a time at most (as they do not have common jumps). In other words, the product R is strictly positive if and only if $R^{(+)}$ and $R^{(-)}$ are strictly positive. Then, one can show that $R^{(+)}$ is a uniformly integrable martingale as an application of Proposition A.1 in Appendix A.

(b) If $\mathbb{Q}^m \in \mathcal{M}(\mathbb{G},\mathbb{P})$, then $R = \mathcal{E}(L)$ is a strictly positive, uniformly integrable $(\mathbb{G},\mathbb{P})$ martingale, where $L = -\int \alpha_s dM_s$. We denote $H_t := \mathbb{1}_{\alpha_t>0}$ and we can write $R^{(-)} = \mathcal{E}_t(\int HdL)$ and $R^{(+)} = \mathcal{E}_t(\int(1-H)dL)$. The result follows as an application of the Theorem A.2 in Appendix A.

□

Another sufficient condition for $\mathbb{Q}^{sup} \in \mathcal{S}up(\mathbb{G},\mathbb{P})$ is the following:

Lemma 3.6. *Suppose that it exists $\widetilde{\mathbb{P}} \in \mathcal{S}up_T(\mathbb{G},\mathbb{P})$ with $d\widetilde{\mathbb{P}}/d\mathbb{P}|_{\mathcal{G}_t} = \mathcal{E}(L)_t$ satisfying $\mathbb{E}^{\mathbb{P}}[e^{\frac{1}{2}\langle L\rangle_T}] < \infty$. Then, $\mathbb{Q}^{sup} \in \mathcal{S}up_T(\mathbb{G},\mathbb{P})$.*

Proof. The supermartingale measure $\widetilde{\mathbb{P}}$ for X can be writen as $d\widetilde{\mathbb{P}}/d\mathbb{P} = \mathcal{E}(L)_T$ on $\mathcal{G}_T$ where L is a $(\mathbb{G},\mathbb{P})$ martingale such that $(\mathcal{E}_{t\wedge T}(L), t \geq 0)$ is a strictly positive uniformly integrable martingale. We denote by ρ the density of $d\langle L, M\rangle$ with respect to $d\langle M\rangle$, i.e., L has the representation: $L = \int \rho dM + N$ with N orthogonal to M (the Kunita-Watanabe decomposition). In order for X to be a $(\mathbb{G},\widetilde{\mathbb{P}})$-supermartingale, necessarily the process:

$$\int_0^t (\alpha_u + \rho_u)\, d\langle M\rangle_u, \quad t \in [0,T]$$

is decreasing, in particular we need:

$$0 \leq \alpha \mathbb{1}_{\{\alpha>0\}} \leq -\rho \mathbb{1}_{\{\alpha>0\}} \quad a.s.$$

By assumption, $\mathcal{E}(L)$ satisfies the Novikov condition. Hence, in view of the above inequality, $R^{(+)}$ is a uniformly integrable martingale satisfying the Novikov condition. □

4. The (H) hypothesis as a no arbitrage condition in presence of short sales restrictions

In this section, we further analyse connections between the existence of $\mathbb{Q}^m$ and $\mathbb{Q}^{sup}$, in particular under the assumption that the (H) hypothesis under some probability measure holds true. Given two filtrations, $\mathbb{F} \subset \mathbb{G}$, the (H) hypothesis, also called the immersion property, is the property that all $\mathbb{F}$ martingales are also $\mathbb{G}$ martingales; the immersion property holds in relation to a reference probability measure. In the financial literature, the (H) hypothesis under an equivalent martingale measure for the asset prices is known as a classical no arbitrage condition whenever asset prices are $\mathbb{F}$ adapted, but agents employ $\mathbb{G}$-adapted strategies. The main result is due to Blanchet-Scalliet and Jeanblanc [6] (see Theorem 4.2 below), further results involving the (H) hypothesis and no arbitrage can be found in [11], and in particular in the case where the filtration $\mathbb{G}$ is obtained from $\mathbb{F}$ via a progressive enlargment with a random time. All these results are investigating the link between (NFLVR) in $\mathbb{G}$ and (H), more precisely, when the $\mathbb{G}$ informed agent are able to sell short. We shall prove that in the case of a continuous price process, the immersion property is also a key condition for understanding (NFLVRS) in $\mathbb{G}$, that is, existence of arbitrages when the insider does not sell short. This result might seem surprising at a first sight: when the insider is not allowed to short-sell, weaker constraints need to be imposed on a price processes to avoid possible arbitrages.

Definition 4.1. *We say that the immersion property, or (H) hypothesis, holds between $\mathbb{F}$ and $\mathbb{G}$ under the probability measure $\mathbb{Q}$ if all $(\mathbb{F},\mathbb{Q})$ local martingales are $(\mathbb{G},\mathbb{Q})$ local martingales. When this property holds, we write $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$. When this property does not hold, we write $\mathbb{F} \overset{\mathbb{Q}}{\not\hookrightarrow} \mathbb{G}$*

More about the immersion of filtrations can be found in [7], [12] or [3]. See also [27], where changes of the probability measure are considered together with the immersion property. For the reader's convenience, we recall the result of Blanchet-Scalliet and Jeanblanc that was mentioned above:

Theorem 4.2. *We suppose that $\mathbb{F}$-(PRP) holds and $\mathcal{M}(\mathbb{G}, \mathbb{P}) \neq \emptyset$. Then, there exists $\widetilde{\mathbb{Q}} \sim \mathbb{P}$ such that $\mathbb{F} \overset{\widetilde{\mathbb{Q}}}{\hookrightarrow} \mathbb{G}$. Moreover, such a $\widetilde{\mathbb{Q}}$ can be chosen as an element of $\mathcal{M}(\mathbb{G}, \mathbb{P})$.*

In our framework, because X is assumed already to be a local martingale for $(\mathbb{F}, \mathbb{P})$, even stronger relations hold:

Theorem 4.3. *We suppose that $\mathbb{F}$-(PRP) holds. Then, the following are equivalent:*

(i) $\mathcal{M}(\mathbb{G}, \mathbb{P}) \neq \emptyset$

(ii) *there exists $\widetilde{\mathbb{Q}} \sim \mathbb{P}$ such that $\mathbb{F} \overset{\widetilde{\mathbb{Q}}}{\hookrightarrow} \mathbb{G}$.*

(iii) *there exists $\mathbb{Q} \sim \mathbb{P}$ such that $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$ and $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_\infty$.*

(iv) *there exists $\widehat{\mathbb{Q}} \in \mathcal{S}up(\mathbb{G}, \mathbb{P})$ such that $\mathbb{F} \overset{\widehat{\mathbb{Q}}}{\hookrightarrow} \mathbb{G}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is Theorem 4.2. The implication (ii) $\Rightarrow$ (i) comes by an application of Proposition 4.4 in [11] which says that whenever there is a $\widetilde{\mathbb{Q}} \sim \mathbb{P}$ satisfying $\mathbb{F} \overset{\widetilde{\mathbb{Q}}}{\hookrightarrow} \mathbb{G}$, there is as well $\mathbb{Q} \sim \mathbb{P}$ satisfying $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$ such that every $(\mathbb{F}, \mathbb{P})$ martingale is a $(\mathbb{G}, \mathbb{Q})$ martingale. In particular, in our setting: X is a $(\mathbb{F}, \mathbb{P})$ martingale and hence a $(\mathbb{G}, \mathbb{Q})$ martingale, that is to say, $\widetilde{\mathbb{Q}} \in \mathcal{M}(\mathbb{G}, \mathbb{P})$ and (i) holds true. But this also means that all $(\mathbb{F}, \mathbb{P})$ martingales are $(\mathbb{F}, \mathbb{Q})$ martingales (as they are a $(\mathbb{G}, \mathbb{Q})$ martingales, $\mathbb{F}$ adapted), that is: $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_\infty$. This proves that (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (ii) is obvious. It remains to prove the tast equivalence. This is also straightforward: (iv) implies (ii) and also (iii) implies (iv), as $\mathbb{Q}$ is martingale measure, so that it is also a supermartingale measure. $\square$

We see that for the insider, as soon as there exists a probability measure equivalent to $\mathbb{P}$ under which the (H) hypothesis holds, there are no arbitrages to undertake, without or even involving short positions. On the other hand, if there is not such an equivalent probability measure, then there are automatically arbitrages, possibly involving short-selling, as $\mathcal{M}(\mathbb{G}, \mathbb{P}) = \emptyset$. Theorem 4.3 tells us that the situations where $\mathcal{M}(\mathbb{G}, \mathbb{P}) = \emptyset$ and $\mathcal{S}up(\mathbb{G}, \mathbb{P}) \neq \emptyset$ are necessarily such that no element of $\mathcal{S}up(\mathbb{G}, \mathbb{P})$ can ensure for the (H) hypothesis to hold, when taken as a reference measure.

Theorem 4.4 below is a result that sheds some additional light on the connections between the $\mathbb{Q}^{sup}$ and $\mathbb{Q}^m$, namely, we provide conditions under which both fail to be probability measures. As we shall only focus on a bounded time horizon $[0, T]$, we introduce the stopped filtrations $\mathbb{F}_T = (\mathcal{F}_{T \wedge t})$ and $\mathbb{G}_T = (\mathcal{G}_{T \wedge t})$, and we suppose that all stochastic processes are also stopped at T without introducing further notation. We also shall assume that $\mathbb{F}_T \overset{\mathbb{P}}{\not\hookrightarrow} \mathbb{G}_T$, that is, the process α is not constantly null, so that our model is not incompatible with the failure of (NFLVR) in the filtration $\mathbb{G}$.

Let us introduce the following set of probability measures:

$$\mathcal{P}_T(\mathbb{F}, \mathbb{G}, \mathbb{P}) := \left\{ \widetilde{\mathbb{Q}} \text{ probability on } (\Omega, \mathcal{G}_T) \;\middle|\; \begin{array}{l} \mathbb{F}_T \overset{\widetilde{\mathbb{Q}}}{\hookrightarrow} \mathbb{G}_T, \\ \mathbb{P} \ll \widetilde{\mathbb{Q}};\ \mathbb{P} \sim \widetilde{\mathbb{Q}} \text{ on } \mathcal{F}_T. \end{array} \right\}.$$

Absolutely continuous, but not equivalent, changes of measure may lead to arbitrage opportunities in the (NFLVR) sense, see e.g. Delbaen and Schachermayer [14], Osterrieder and Rheinländer [29], Ruf and Runggaldier [31] and Chau and Tankov [8]. Here, as we have taken as a reference measure an equivalent probability measure for the filtration $\mathbb{F}$, we need to go the reverse direction, that is, to assume that $\mathbb{P}$ is absolutely continuous to a class of measures (that may include the probability under which the insider observes the market price evolution). Below, we are going to assume the existence of a measure generalising the minimal martingale measure, in that it is not necessarily absolutely continuous to $\mathbb{P}$.

Theorem 4.4. *We suppose that (C) and $\mathbb{F}$-(PRP) hold. Suppose that on $(\Omega, \mathcal{G})$ there exists a probability measure $\widetilde{\mathbb{Q}} \in \mathcal{P}_T(\mathbb{F}, \mathbb{G}, \mathbb{P})$ and such that all $(\mathbb{G}, \mathbb{P})$ martingales orthogonal to M are martingales under $\widetilde{\mathbb{Q}}$. Then, the following hold:*

(a) *If $\mathbb{P}$ and $\widetilde{\mathbb{Q}}$ are not equivalent on $\mathcal{G}_T$, then $\mathbb{Q}^m \notin \mathcal{M}_T(\mathbb{G}, \mathbb{P})$ and $\mathbb{Q}^{sup} \notin \mathcal{S}up_T(\mathbb{G}, \mathbb{P})$.*

(b) *If $\mathbb{P} \sim \widetilde{\mathbb{Q}}$ on $\mathcal{G}_T$ and $D_T \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$, then $\mathbb{Q}^{\text{sup}} \in \mathcal{S}up_T(\mathbb{G}, \mathbb{P})$ and $\mathbb{Q}^m \in \mathcal{M}_T(\mathbb{G}, \mathbb{P})$.*

In Theorem 4.4, if $\mathbb{P}$ and $\mathbb{Q}$ are not equivalent on $\mathcal{G}_T$, this does not exclude a priori the existence of a probability $\widetilde{\mathbb{Q}}$ satisfying the properties in Theorem 4.3 (iii), and we cannot conclude $\mathcal{M}_T(\mathbb{G}, \mathbb{P}) = \emptyset$. The proof of this theorem is postponed at the end of this section.

First, let us prove some intermediary result:

Lemma 4.5. *We assume that $\mathbb{F}$-(PRP) holds. Suppose that there exists $\widetilde{\mathbb{Q}}$ satisfying the conditions from Theorem 4.4. Then, there is also $\mathbb{Q} \sim \widetilde{\mathbb{Q}}$ that satisfies:*

(i) $\mathbb{Q} \in \mathcal{P}_T(\mathbb{F}, \mathbb{G}, \mathbb{P})$,

(ii) *The process $\mathbb{E}^{\mathbb{Q}}[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{G}_{t\wedge T}] = D_t$ is a nonnegative $(\mathbb{G}_T, \mathbb{Q})$ martingale having the representation:*

$$D_t = 1 + \int_0^t \varphi_s dX_s, \tag{6}$$

where φ is a $\mathbb{G}_T$ predictable process having a null $(\mathbb{F}_T, \mathbb{Q})$ optional projection.

Proof. If $\widetilde{\mathbb{Q}}$ is as in Proposition 4.4, then there is a nonnegative $(\mathbb{G}_T, \widetilde{\mathbb{Q}})$ martingale E with $d\mathbb{P} = E_T \cdot \widetilde{\mathbb{Q}}$ on $\mathcal{G}_T$ and a strictly positive $(\mathbb{F}_T, \widetilde{\mathbb{Q}})$ martingale e with $d\mathbb{P} = e_T \cdot \widetilde{\mathbb{Q}}$ on $\mathcal{F}_T$. Because $\mathbb{F}$-(PRP) holds and $\widetilde{\mathbb{Q}} \sim \mathbb{P}$ on $\mathcal{F}_T$, it follows that in the filtration $\mathbb{F}$ the $\widetilde{\mathbb{Q}}$ martingales have the predictable representation property with respect to the continuous martingale $\widetilde{X} = X - \int \frac{d\langle e, X\rangle}{e}$ (see [22], Theorem 13.12]). Then, there exists an $\mathbb{F}$ predictable process h so that $e = \mathcal{E}(\int h d\widetilde{X})$. As (H) holds under $\widetilde{\mathbb{Q}}$, $\widetilde{X}$ is also a $\widetilde{\mathbb{Q}}$ martingale in the larger filtration $\mathbb{G}_T$. Then, e is also a $(\mathbb{G}_T, \widetilde{\mathbb{Q}})$ martingale and E has the representation $E = \mathcal{E}(N + \int H d\widetilde{X})$ with N a $(\mathbb{G}_T, \widetilde{\mathbb{Q}})$ local martingale orthogonal to $\widetilde{X}$, this representation of E being $\widetilde{\mathbb{Q}} - a.s.$ unique on the set $[0, T^0]$ with $T^0 = \inf\{t \geq 0 : E_t = 0\}$. We observe that $N_{\cdot \wedge T_0} \equiv 0$ since otherwise there are $(\mathbb{G}_T, \mathbb{P})$ local martingale $N - \langle N \rangle$, orthogonal to M is not a local martingale anymore under $\widetilde{\mathbb{Q}}$. So, we have the representation $E = \mathcal{E}(\int H d\widetilde{X})$.

We introduce $\mathbb{Q}$ via: $d\mathbb{Q} = e_T \cdot d\widetilde{\mathbb{Q}}$ on $\mathcal{G}_T$. We now prove that $\mathbb{Q}$ fulfils the required properties. We can easily check that the following holds for $t \in [0, T]$:

$$D_t := \frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{G}_t} = \frac{d\mathbb{P}}{d\widetilde{\mathbb{Q}}}\Big|_{\mathcal{G}_t} \times \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{G}_t} = \frac{E_t}{e_t} = \mathcal{E}_t\left(\int (H - h) dX\right)$$

and

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{d\mathbb{P}}{d\widetilde{\mathbb{Q}}}\Big|_{\mathcal{F}_t} \times \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \frac{e_T}{e_T} = 1.$$

As $\mathbb{E}_{\mathbb{Q}}[D_t|\mathcal{F}_t] = \frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t} = 1$ we have that D is a true martingale and it is nonnegative (product on nonnegative processes). It follows that $\mathbb{P} \ll \mathbb{Q}$ and $\mathbb{P} = \mathbb{Q}$ on $\mathcal{F}_T$. Therefore, D has the representation (18) with $\varphi_t = D_{t-}(H_t - h_t)$ and the property $\mathbb{E}_{\mathbb{Q}}[\varphi_t|\mathcal{F}_t] = 0$ for all $t \in [0,T]$ is resulting from the required property $\mathbb{E}_{\mathbb{Q}}[D_t|\mathcal{F}_t] = 1 + \mathbb{E}_{\mathbb{Q}}[\int_0^t \varphi_s dX_s|\mathcal{F}_t] = 1$ and the use of Proposition 3 in Brémaud and Yor [7]. □

In order to prove Theorem 4.4, we abandon temporarily the framework introduced Section 2 to take as reference the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. We suppose that (H) hypothesis holds under $\mathbb{Q}$. The aim is to recover $\mathbb{P}$ form $\mathbb{Q}$ with the help of a Girsanov transformation. The key properties of our initial model under $\mathbb{P}$, in particular R and $R^{(+)}$ being or not strict local martingales, will appear as resulting from whether $\mathbb{P}$ is actually equivalent to $\mathbb{Q}$ or only absolutely continuous (see Proposition 4.6 below). We will start with two filtrations, labeled $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)$ and $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_t)$ with $\tilde{\mathbb{G}}$ being strictly larger than $\tilde{\mathbb{F}}$. These filtrations satisfy the usual assumptions, in particular are augmented with the null sets of $\mathbb{Q}$. The filtrations $\mathbb{F}$ and $\mathbb{G}$ will be defined below, as further completions with the null sets of $\mathbb{P}$.

The assumptions that are going to be used below are the following:

(A1) The filtration $\tilde{\mathbb{F}}$ is the natural filtration of a $\mathbb{Q}$-Brownian motion B.

(A2) The filtrations $\tilde{\mathbb{F}}$ and $\tilde{\mathbb{G}}$ satisfy $\tilde{\mathbb{F}} \subset \tilde{\mathbb{G}}$ and $\tilde{\mathbb{F}} \overset{\mathbb{Q}}{\hookrightarrow} \tilde{\mathbb{G}}$.

(A3) The time horizon is $[0,T]$, T constant and $\mathcal{F} = \mathcal{G}_T$ (so that all probabilities will be defined on $\mathcal{G}_T$ directly).

(A4) There is a nonnegative $(\tilde{\mathbb{G}}, \mathbb{Q})$ martingale having the representation:

$$D_t = 1 + \int_0^t G_s dB_s, \tag{7}$$

where G is a $\mathbb{G}$ predictable process satisfying $\mathbb{E}^{\mathbb{Q}}[G_t|\tilde{\mathcal{F}}_t] = 0$ for all $t \in [0,T]$.

We now define the probability $\mathbb{P}$ as:

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\tilde{\mathcal{G}}_T} = D_T, \tag{8}$$

where D is the martingale defined in (18).

Proposition 4.6. *We consider the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with the assumptions (A1)-(A4) and let $\mathbb{P}$ be defined as in (8). The filtration $\mathbb{F}$ is now defined as the usual augmentation of $\tilde{\mathbb{F}}$ with the $\mathbb{P}$ null sets and similarly, the filtration $\mathbb{G}$ is defined as the usual augmentation of $\tilde{\mathbb{G}}$ with the $\mathbb{P}$ null sets. Let X be an $\mathbb{F}$ adapted stochastic process that is a $\mathbb{Q}$ local martingale. The following hold:*

(a) *The process X is a $(\mathbb{F}, \mathbb{P})$ local martingale and it is a $(\mathbb{G}, \mathbb{P})$ semimartingale with the Doob Meyer decomposition:*

$$X_t = X_0 + M_t + \int_0^t \alpha_s d\langle M\rangle_s \tag{9}$$

for a $(\mathbb{G}, \mathbb{P})$ martingale M and a $\mathbb{G}$-predictable process α having a null $(\mathbb{F}, \mathbb{P})$ optional projection. Thus, X satisfies the assumptions from Section 2 under $\mathbb{P}$.

Define $R := \mathcal{E}\left(-\int_0^\cdot \alpha_u dM_u\right)$ and $R^{(+)} := \mathcal{E}\left(-\int_0^\cdot \alpha_u^+ dM_u\right)$.

(b) *If* $\mathbb{P} \sim \mathbb{Q}$ *and* $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{P}}{d\mathbb{Q}}] < \infty$*, then, both* R *and* $R^{(+)}$ *are true* $(\mathbb{G}, \mathbb{P})$*-martingales.*

(c) *If* $\mathbb{P} \ll \mathbb{Q}$ *but* $\mathbb{P}$ *is not equivalent to* $\mathbb{Q}$*, then both* R *and* $R^{(+)}$ *are strict* $(\mathbb{G}, \mathbb{P})$*-local martingales.*

Proof. Within this proof, but also in the proofs of the other results in this section, we shall use the following:

Notation. For any $\mathbb{G}$-adapted process Y, we denote ${}^{o,\mathbb{Q}}Y$ its optional projection onto the filtration $\mathbb{F}$ under the probability $\mathbb{Q}$; whenever Y is increasing, $Y^{o,\mathbb{Q}}$ stands for its $(\mathbb{F}, \mathbb{Q})$-dual optional projection.

Proof of claim in (a). The process X being an $(\mathbb{F}, \mathbb{Q})$-local martingale, there is an $\mathbb{F}$-predictable process F such that

$$X = X_0 + \int F dB. \tag{10}$$

We shall prove that the assumption (A4) implies:

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_T} = 1 \tag{11}$$

which then implies that all $(\mathbb{F}, \mathbb{Q})$-local martingales remain $(\mathbb{F}, \mathbb{P})$ local martingales, in particular X. Also, we can remark that $\mathbb{F} = \widetilde{\mathbb{F}}$.

The density process $(\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t})_{t\in[0,T]}$ is given by the $(\mathbb{F}, \mathbb{Q})$ optional projection of the process D. The Brownian motion B being an $(\mathbb{F}, \mathbb{Q})$ and a $(\mathbb{G}, \mathbb{Q})$ local martingale (by assumption (A2)), we obtain (see [7]):

$$\frac{d\mathbb{P}}{d\mathbb{Q}}|_{\mathcal{F}_t} = {}^{o,\mathbb{Q}}D_t = 1 + \int_0^t {}^{o,\mathbb{Q}}G_s dB_s.$$

From assumption (A4), ${}^{o,\mathbb{Q}}G \equiv 0$ and we obtain ${}^{o,\mathbb{Q}}D \equiv 1$, hence (11) holds true. Consequently, X is also a $(\mathbb{F}, \mathbb{P})$-local martingale (by using Girsanov's theorem in the filtration $\mathbb{F}$).

In the larger filtration $\mathbb{G}$ however, using the Lenglart's extenstion of the Girsanov's theorem, we obtain that the following is a martingale under $\mathbb{P}$:

$$M := X - X_0 - \int_0^t \frac{d\langle X, D\rangle_s}{D_s} = X - X_0 - \int_0^t \frac{G_s F_s}{D_s} ds.$$

Because from (10) $\langle X\rangle = \int F^2 ds$, we can write

$$X = X_0 + M + \int_0^t \alpha_s d\langle M, M\rangle_s$$

with

$$\alpha_t := \mathbb{1}_{F_t\neq 0}\mathbb{1}_{D_t\neq 0}\frac{G_t}{D_t F_t}. \tag{12}$$

For all $t \geq 0$:

$$\mathbb{E}_{\mathbb{P}}[\alpha_t|\mathcal{F}_t] = \mathbb{1}_{F_t\neq 0}\mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{D_t\neq 0}\frac{G_t}{D_t}|\mathcal{F}_t\right]\frac{1}{F_t} = \mathbb{1}_{F_t\neq 0}\frac{\mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\frac{G_t}{D_t}|\mathcal{F}_t\right]}{\mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{F}_t\right]F_t}$$

$$= \mathbb{1}_{F_t\neq 0}\frac{\mathbb{E}_{\mathbb{Q}}[G_t|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}}[D_t|\mathcal{F}_t]F_t} = 0,$$

thus, α has a null $(\mathbb{F}, \mathbb{P})$ optional projection. The claim in (a) is thus proved.

Proof of claims (b) and (c). We write D as product of orthogonal $(\mathbb{G}, \mathbb{Q})$ local martingales in two different ways. First, we write $D = LZ$ with

$$\begin{aligned} dL_t &= L_t \frac{G_t}{D_t} \mathbb{1}_{\{F_t \neq 0\} \cap \{D_t > 0\}} dB_t \\ &= L_t \alpha_t F_t dB_t \\ L_0 &= 1 \end{aligned}$$

and

$$\begin{aligned} dZ_t &= Z_t \frac{G_t}{D_t} \mathbb{1}_{\{F_t = 0\} \cap \{D_t > 0\}} dB_t \\ Z_0 &= 1 \end{aligned}$$

Second, we write $D = L^{(+)} Z^{(-)}$ with

$$\begin{aligned} dL_t^{(+)} &= L_t^{(+)} \frac{(G_t)^+}{D_t} \mathbb{1}_{\{F_t \neq 0\} \cap \{D_t > 0\}} dB_t \\ &= L_t^{(+)} (\alpha_t)^+ F_t dB_t \\ L_0^{(+)} &= 1 \end{aligned}$$

and

$$\begin{aligned} dZ_t^{(-)} &= Z_t^{(-)} \frac{\mathbb{1}_{\{D_t > 0\}}}{D_t} \left((G_t)^+ \mathbb{1}_{\{F_t = 0\}} - (G_t)^- \right) dB_t \\ Z_0^{(-)} &= 1. \end{aligned}$$

The following relations can be easily verified (using Ito's formula and the expression of the processes X and α in (9) resp (12)):

$$\begin{aligned} R_t &= \frac{1}{L_t} \\ R^{(+)} &= \frac{1}{L_t^{(+)}}. \end{aligned}$$

We denote:

$$\begin{aligned} T_0 &:= \inf\{t \geq 0 : D_t = 0\} \wedge T \\ \tau &:= \inf\{t \geq 0 : L_t = 0\} \wedge T \\ \tau^{(+)} &:= \inf\{t \geq 0 : L_t^{(+)} = 0\} \wedge T. \end{aligned}$$

We notice that the processes L, $L^{(+)}$, Z and $Z^{(-)}$ are such that the measure $d\langle L \rangle$ is singular with respect to the measure $d\langle Z \rangle$ and also the measure $d\langle L^{(+)} \rangle$ is singular with respect to the measure $d\langle Z^{(-)} \rangle$. These local martingales being continuous, their constancy intervals are determined by the constancy intervals of their sharp bracket; hence L and Z do not co-move and $L^{(-)}$ and $Z^{(+)}$ do not co-move. Additionally the process D, a nonnegative martingale, is absorbed at the level 0, once this level attained, and consequently the processes L, $L^{(+)}$, Z and $Z^{(-)}$ are also stopped at

T_0. This shows that the following equalities hold $\mathbb{Q}$ a.s.:

$$\{\tau < T\} = \{L_T = 0\} = \{Z_T > 0\} \cap \{T_0 < T\}; \tag{13}$$

$$\{\tau^{(+)} < T\} = \{L_T^{(+)} = 0\} = \{Z_T^{(-)} > 0\} \cap \{T_0 < T\}. \tag{14}$$

We know that $R_t = \frac{1}{L_t}$. Therefore:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left[R_T\right] &= \mathbb{E}_{\mathbb{Q}}\left[D_T \frac{1}{L_T}\right] = \mathbb{E}_{\mathbb{Q}}\left[D_T \frac{1}{L_T} \mathbb{1}_{L_T > 0}\right] = \mathbb{E}_{\mathbb{Q}}\left[Z_T \mathbb{1}_{L_T > 0}\right] \\ &= \mathbb{E}_{\mathbb{Q}}[Z_T] - \mathbb{E}_{\mathbb{Q}}\left[Z_T \mathbb{1}_{L_T = 0}\right] \end{aligned}$$

and from this we deduce that

- if $\mathbb{Q}(\tau < T) > 0$ we have $\mathbb{E}_{\mathbb{P}}\left[R_T\right] < 1$ i.e., R is a strict $(\mathbb{G}, \mathbb{P})$-local martingale. The strict inequality is justified using (13). Indeed, these equalities imply that

$$\mathbb{Q}(\tau < T) = \mathbb{Q}(L_T = 0) > 0$$

  and $\{L_T = 0\} \subset \{Z_T > 0\}$ which leads to $\mathbb{E}_{\mathbb{Q}}\left[Z_T \mathbb{1}_{L_T = 0}\right] > 0$, hence the result.
- if $\mathbb{Q}(\tau < T) = 0$ and Z a true $\mathbb{Q}$ martingale, we have $\mathbb{E}_{\mathbb{P}}\left[R_T\right] = 1$ i.e., R is a true $(\mathbb{G}, \mathbb{P})$-martingale.

Identical steps as above for $R = \frac{1}{L}$ can be used now for $R^{(+)} = \frac{1}{L^{(+)}}$. Indeed, the characterisation of $\tau^{(+)}$ in (14), can be used to prove the following:

- if $\mathbb{Q}(\tau^{(+)} < T) > 0$ we have $\mathbb{E}_{\mathbb{P}}\left[R_T^{(+)}\right] < 1$ i.e., $R^{(+)}$ is a strict $(\mathbb{G}, \mathbb{P})$-local martingale.
- if $\mathbb{Q}(\tau^{(+)} < T) = 0$ and $Z^{(-)}$ a true $\mathbb{Q}$ martingale, we have $\mathbb{E}_{\mathbb{P}}\left[R_T^{(+)}\right] = 1$ i.e., $R^{(+)}$ is a true $(\mathbb{G}, \mathbb{P})$-martingale.

The claims (b) and (c) can now be proved using the above analysis and the fact that $\mathbb{Q}(\tau^{(+)} < T) > 0$ if and only if $\mathbb{Q}(\tau < T) > 0$. For more clarity, the poof of this result is stated as a separate result below (Lemma 4.7).

In particular, for (b): The condition $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{P}}{d\mathbb{Q}}] < \infty$ is the same as the square integrability of D under $\mathbb{Q}$. As D is by assumption a square integrable $(\mathbb{G}, \mathbb{Q})$-martingale, using Proposition A.1, we deduce that, L, $L^{(+)}$, Z and $Z^{(+)}$ are strictly positive and uniformly integrable $(\mathbb{G}, \mathbb{Q})$-martingales. We use Lemma 4.7 below and the above analysis to conclude. For (c) we see that there is no need to have Z and $Z^{(-)}$ be true $\mathbb{Q}$ martingales, so that no more constraints on the process D are needed. $\square$

Lemma 4.7. *The following are equivalent: (i)* $\mathbb{Q}(T_0 < T) > 0$*; (ii)* $\mathbb{Q}(\tau^{(+)} < T) > 0$*; (iii)* $\mathbb{Q}(\tau < T) > 0$.

Proof. To begin with, we give an alternative characterisation of the sets $\{\tau < T\}$ and $\{\tau^{(+)} < T\}$, that uses the fact that the martingale L is constant on the set $\{(\omega, t) : F_t(\omega) = 0\}$; while $L^{(+)}$ is constant on the set $\{(\omega, t) : F_t(\omega) = 0, G_t(\omega) \leq 0\}$ (these properties can be easily seen from an inspection of their quadratic variation).

$$\begin{aligned} \{\tau < T\} &= \{T_0 = \tau\} \cap \{T_0 < T\} = \{F_{T_0} \neq 0\} \cap \{T_0 < T\}; \\ \{\tau^{(+)} < T\} &= \{T_0 = \tau^{(+)}\} \cap \{T_0 < T\} \\ &= \{F_{T_0} \neq 0\} \cap \{G_{T_0} > 0\} \cap \{T_0 < T\}. \end{aligned}$$

Therefore, denoting by $A := (\mathbb{1}_{T_0<t})_{t\in[0,T]}$:

$$\mathbb{Q}(\tau^{(+)} < T) = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{F_{T_0}\neq 0}\mathbb{1}_{G_{T_0}>0}\mathbb{1}_{T_0<T}] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \mathbb{1}_{F_t\neq 0}\mathbb{1}_{G_t>0}dA_t\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \mathbb{1}_{F_t\neq 0}\mathbb{Q}(G_t>0|\mathcal{F}_t)dA_t^{o,\mathbb{Q}}\right] \tag{15}$$

and

$$\mathbb{Q}(\tau < T) = \mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{F_{T_0}\neq 0}\mathbb{1}_{T_0<T}] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \mathbb{1}_{F_t\neq 0}dA_t\right].$$

We notice that the measure dA_t does not charge the set $\{(\omega,t) : G_t(\omega) = 0\}$, that is:

$$dA_t = \mathbb{1}_{G_t\neq 0}dA_t.$$

This property can be seen as following from Dubins-Schwarz theorem, which allows to write

$$T_0 = \inf\left\{t : W_{\int_0^t G_s^2 ds} = 0\right\}$$

where W is a Brownian motion (alternatively, the hitting time of 0 by D cannot occur on the constancy intervals of the process D).

Hence:

$$\mathbb{Q}(\tau < T) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \mathbb{1}_{F_t\neq 0}\mathbb{1}_{G_t\neq 0}dA_t\right]$$
$$= \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \mathbb{1}_{F_t\neq 0}\mathbb{Q}(G_t\neq 0|\mathcal{F}_t)dA_t^{o,\mathbb{Q}}\right] \tag{16}$$

We remark that because $\mathbb{E}_{\mathbb{Q}}[G_t|\mathcal{F}_t] = 0$, we have that $\mathbb{Q}(G_t < 0|\mathcal{F}_t) > 0$ if and only if $\mathbb{Q}(G_t > 0|\mathcal{F}_t) < 0$. Therefore:

$$\mathbb{Q}(G_t \neq 0|\mathcal{F}_t) > 0 \text{ if and only if } \mathbb{Q}(G_t > 0|\mathcal{F}_t) > 0. \tag{17}$$

From (15), we see that $\mathbb{Q}(\tau^{(+)} < T) > 0$ if and only if the measure $dA_t^{o,\mathbb{Q}}$ charges a subset of

$$\mathcal{A}^{(+)} := \{(\omega,t)\in\Omega\times[0,T] | F_t(\omega)\neq 0; \mathbb{Q}(G_t>0|\mathcal{F}_t)(\omega) > 0\}.$$

From (16), we see that $\mathbb{Q}(\tau < T) > 0$ if and only if the measure $dA_t^{o,\mathbb{Q}}$ charges a subset of

$$\mathcal{A} := \{(\omega,t)\in\Omega\times[0,T] | F_t(\omega)\neq 0; \mathbb{Q}(G_t\neq 0|\mathcal{F}_t)(\omega) > 0\}.$$

But because of (17) we have $\mathcal{A}^{(+)} = \mathcal{A}$. Thus, we have proved that $\mathbb{Q}(\tau^{(+)} < T) > 0$ if and only if $\mathbb{Q}(\tau < T) > 0$. Obviously, each one of these implies $\mathbb{Q}(T_0 < T) > 0$. □

Remark 4.8. *The results in Proposition 4.6 remain true if we replace the Brownian motion B by the continuous martingale X having the predictable representation property in a filtration $\widetilde{\mathbb{F}}$ under $\mathbb{Q}$. More exactly, the statements in Proposition 4.6 are true if we can replace the assumptions (A1), (A2), (A3) and (A4) with the assumptions (A1'), (A2), (A3) and (A4'), where*

(A1') *There is a continuous martingale X having the predictable representation property with respect to $(\tilde{\mathbb{F}}, \mathbb{Q})$.*

(A4') *There is a nonnegative* $(\tilde{\mathbb{G}}, \mathbb{Q})$ *martingale having the representation:*

$$D_t = 1 + \int_0^t G_s dX_s, \tag{18}$$

where G *is a* $\mathbb{G}$ *predictable process satisfying* $\mathbb{E}^{\mathbb{Q}}[G_t|\tilde{\mathcal{F}}_t] = 0$ *for all* $t \in [0,T]$.

The proof is identical; we simply have to consider that $F \equiv 1$, *where* F *is the process appearing in the proof of Proposition 4.6, see equation (10) and to replace* B *by* X *everywhere.*

Proof. (Theorem 4.4) We consider the setting of Section 2, that is, the probability space is $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that (C) and $\mathbb{F}$-(PRP) hold. Suppose that on $(\Omega, \mathcal{F})$ there exists a probability measure $\widetilde{\mathbb{Q}}$ such that $\widetilde{\mathbb{Q}} \in \mathcal{P}_T(\mathbb{F}, \mathbb{G}, \mathbb{P})$ and such that all $(\mathbb{G}, \mathbb{P})$ martingales orthogonal to M are martingales under $\widetilde{\mathbb{Q}}$. By Lemma 4.5, there is also a probability $\mathbb{Q} \in \mathcal{P}_T(\mathbb{F}, \mathbb{G}, \mathbb{P})$, such that $D_t := \mathbb{E}^{\mathbb{Q}}[\frac{d\mathbb{P}}{d\mathbb{Q}}|\mathcal{G}_{t\wedge T}]$ has the representation introduced in (18). Using the Remark 4.8 and Proposition 4.6, we deduce that:

- If $\mathbb{P} \sim \mathbb{Q}$ and $D_T \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$, then, both R and $R^{(+)}$ are true $(\mathbb{G}, \mathbb{P})$ martingales. This proves (b).
- If $\mathbb{P} \ll \mathbb{Q}$ but not equivalent, then, both R and $R^{(+)}$ are strict $(\mathbb{G}, \mathbb{P})$ local martingales. This proves (a).

□

Appendix A. Some useful results on the product of strictly positive, orthogonal local martingales

Given a uniformly integrable martingale M, that can be factorised as $M = UZ$ with strictly positive local martingales martingales U and Z, it is of importance to know if and when U and Z are as well uniformly integrable martingales. The question has raised already much attention it is connected to the existence of the minimal martingale measure. Delbaen and Shachermayer [16] have shown that in general, it is not the case that U and Z are both uniformly integrable, even when one considers continuous processes. Below, we emphasise two particular situations where the implication that U and Z are martingales holds true, relevant in our search for equivalent supermartingale measures, in particular $\mathbb{Q}^{sup}$.

In this appendix we use a filtered probability space that we label $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{P})$ and all stochastic processes are considered adapted to it. Also, $\mathbb{E}$ now stands for $\mathbb{E}^{\mathbb{P}}$, as no confusion can occur.

Proposition A.1. *Suppose that* U *and* Z *are two strictly positive local martingales with* $\Delta U \Delta Z \equiv 0$ *and such that the product* $M := UZ$ *is a square integrable martingale. Then* U *and* Z *are uniformly integrable martingales.*

Proof. Without loss of generality we assume $U_0 = Z_0 = 1$.

First, let us notice that because U and Z are positive local martingales, they converge and $\mathbb{E}[U_\infty] \leq 1, \mathbb{E}[Z_\infty] \leq 1$.

Let us define:

$$\theta^n := \inf\{t | U_t \leq 1/n\} = \inf\{t | Z_t \geq nM_t\}, n \geq 1.$$

We show first that (θ^n) is a localising sequence for Z, that transforms the stopped process $Z^n := Z_{\theta^n \wedge \cdot}$ in a square integrable martingale. It is sufficient to notice:

$$0 < Z_t^n \leq n \max\{M_{t-}^n, M_t^n\}.$$

The inequality above is obvious on the set $\{t < \theta^n\}$. We justify now the last inequality at $t = \theta^n < \infty$ as follows:

- If $\Delta M_{\theta^n} = 0$, then $\Delta Z_{\theta^n} = 0$, hence $Z_{\theta^n} \leq nM_{\theta^n}$ (with equality on $\{\theta^n < \infty\}$).
- If $\Delta M_{\theta^n} \neq 0$ and $\Delta Z_{\theta^n} = 0$, then necessarily $\Delta M_{\theta^n} < 0$ by definition of θ^n and hence $nM_{\theta^n} \leq Z_{\theta^n} \leq nM_{\theta^n-}$ (the process Z was below nM before the negative jump of M at θ_n).
- If $\Delta M_{\theta^n} \neq 0$, $\Delta Z_{\theta^n} \neq 0$, and $\Delta U_{\theta^n} = 0$, i.e., $U_{\theta^n} = 1/n$, we have: $\Delta M_{\theta^n} = \frac{1}{n}\Delta Z_{\theta^n}$, i.e., $\Delta Z_{\theta^n} = n\Delta M_{\theta^n}$. Hence, $Z_{\theta^n} < nM_{\theta^n-} + \Delta Z_{\theta^n} = n(M_{\theta^n-} + \Delta M_{\theta^n}) = nM_{\theta^n}$ (hence this configuration cannot occur, because in contradiction with the definition of θ^n).
- Finally, notice that the case $\Delta M_{\theta^n} \neq 0$, $\Delta Z_{\theta^n} \neq 0$, and $\Delta U_{\theta^n} \neq 0$ is excluded by the condition $\Delta U \Delta Z \equiv 0$.

This proves that Z is a locally square integrable local martingale martingale and so is U, by a symmetric argument. We denote by $\mathcal{Z}$ the stable subset of locally square integrable local martingales generated by the stochastic integrals $\int H_s dZ_s$ and by $\mathcal{Z}^\perp$ the subspace orthogonal to $\mathcal{Z}$. If follows that $U \in \mathcal{Z}^\perp$. For any $t \geq 0$, let $\mathcal{H}_t$ be the $\mathbb{P}$ complete σ-field generated by class of random variables N_t with N being a square integrable element of $\mathcal{Z}^\perp$. Thus, we have a filtration $(\mathcal{H}_t)$ that satisfies the (H) hypothesis, that is, all $(\mathcal{H}_t)$ martingales are $\mathbb{F}$ martingales (see Brémaud and Yor [7], Theorem 3 (3) which applies to our construction). Also, we have that U_t is $\mathcal{H}_t$ measurable (this is true for any N_t where N is square integrable element of $\mathcal{Z}^\perp$ and by using a localisation argument, for U_t as well).

We remark that θ^n is a $(\mathcal{H}_t)$ stopping time (as is a hitting time by U of a non random level $1/n$). As $Z_{t\wedge\theta^n}$ is a square integrable martingale orthogonal to all $(\mathcal{H}_t)$ square integrable martingales, and the hypothesis (H) holds, we have that ${}^oZ = 1$ and consequently $\mathbb{E}[Z_{\theta^n}|\mathcal{H}_{\theta^n}] = Z_0 = 1$. We obtain:

$$1 = \mathbb{E}[M_{\theta^n}] = \mathbb{E}[U_{\theta^n} Z_{\theta^n}] = \mathbb{E}\left[U_{\theta^n}\mathbb{E}[Z_{\theta^n}|\mathcal{H}_{\theta^n}]\right] = \mathbb{E}\left[U_{\theta^n}\right],$$

and therefore θ^n is also a localising sequence for U. Further, for $n \geq 1$:

$$\begin{aligned} 1 = \mathbb{E}[U_{\theta^n}] =& \mathbb{E}[U_\infty \mathbb{1}_{\theta^n=\infty} + U_{\theta^n}\mathbb{1}_{\theta^n<\infty}] \\ \leq& \mathbb{E}[U_\infty \mathbb{1}_{\theta^n=\infty} + 1/n\mathbb{1}_{\theta^n<\infty}] \\ \leq& \mathbb{E}[U_\infty \mathbb{1}_{\theta^n=\infty}] + \mathbb{P}[\theta^n < \infty]. \end{aligned}$$

It follows:

$$\mathbb{E}[(U_0 - U_\infty)\mathbb{1}_{\theta^n=\infty}] \leq 0.$$

Since $|U_0 - U_\infty|\mathbb{1}_{\theta^n=\infty} \leq |U_0 - U_\infty| \in \mathbb{L}^1$, by dominated convergence, $\mathbb{E}[(U_0 - U_\infty)] \leq 0$. But as a supermartingale, we also have $\mathbb{E}[(U_0 - U_\infty)] \geq 0$. Hence U is a uniformly integrable martingale.

The roles of U and Z being interchangeable, the same reasoning can be made to show that Z is a uniformly integrable martingale.

□

Lemma A.2. *Suppose that $M = \mathcal{E}(L)$ is a strictly positive and uniformly integrable martingale. We consider a càdlàg predictable process H with state space $\{0,1\}$ and $U := \mathcal{E}(\int HdL)$ and $Z := \mathcal{E}(\int(1-H)dL)$. Then, if H has a bounded number of transitions, then both U and Z are uniformly integrable.*

Proof. Without loss of generality we shall assume hereafter that $H_0 = 0$.

Let $T^0 = 0$ and for $k \geq 1$, $T^k := \inf\{t \geq T^{k-1} \; |H_t \neq H_{T^{k-1}} \; \}$ i.e., the successive jumping times of the process H, and let (N_t) be the counting process associated with (T^k). We denote $T^\infty = \lim_{k\to\infty} T^k$. Also we denote:

$$M^{(k)} := \mathcal{E}(L_{T^k \wedge .} - L_{T^{k-1}\wedge .}), \quad k = 1, 2, ...$$

We notice that $M^{(k)}$ are orthogonal local martingales and:

$$M = \Pi_{k\geq 1} M^{(k)} \quad U = \Pi_{k\geq 1} M^{(2k)}, \quad Z = \Pi_{k\geq 1} M^{(2k-1)}.$$

Also $M^{(1)} = M_{T^1\wedge .}$ is a uniformly integrable martingale. So is every $M^{(k)}$, $k > 1$:

$$\mathbb{E}[M_\infty^{(k+1)}] = \mathbb{E}\left[\mathbb{E}(M_\infty^{(k+1)}|\mathcal{G}_{T^k})\right] = \mathbb{E}\left[\mathbb{E}(M_{T^{k+1}}|\mathcal{G}_{T^k})\frac{1}{M_{T^k}}\right] = 1. \tag{19}$$

We will now prove the stated properties for U; identical arguments hold for Z. We denote

$$U_\infty := \lim_{t\to\infty} U_t = U_{T^\infty}.$$

We notice that $U_{T^{2k}} = U_{T^{2k-2}} M_\infty^{(2k)}$ and $\mathbb{E}[U_{T^{2k}}|\mathcal{F}_{T_{2k-2}}] = U_{T^{2k-2}} M_\infty^{(2k)}$, therefore, for all $k \geq 1$:

$$\mathbb{E}[U_{T^{2k}}] = \mathbb{E}[M_\infty^{(2k)} U_{T^{2k-2}}] = \mathbb{E}[U_{T^{2k-2}}] = \cdots = U_0 = 1.$$

This proves that if $T^\infty = +\infty \; a.s.$, i.e. (N_t) is finite, then (T^{2k}) is a localising sequence for U.

Therefore, whenever (N_t) is bounded, then U is a uniformly integrable martingale. $\square$

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